Chapter 9: Integration

Exercise 9h

① Let $I = \int x \cdot \cos x \, dx$

Let $u = x$, $\therefore \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos x$, $\therefore v = \sin x$

So $I = x \cdot \sin x - \int 1 \cdot \sin x \, dx$

$= x \sin x - (-\cos x) + c$

$= x \cdot \sin x + \cos x + c$

② Let $I = \int x^2 \cdot e^x \, dx$

Let $u_1 = x^2$, $\therefore \frac{du_1}{dx} = 2x$

$\frac{dv_1}{dx} = e^x$, $\therefore v_1 = e^x$

So $I = x^2 e^x - \int 2x e^x \, dx$

Now let $u_2 = x$, $\therefore \frac{du_2}{dx} = 1$

& $\frac{dv_2}{dx} = e^x$, $\therefore v = e^x$

So $I = x^2 e^x - 2\left[xe^x - \int e^x \, dx\right]$

$= x^2 e^x - 2xe^x + 2\int e^x \, dx$

$= x^2 e^x - 2xe^x + 2e^x + c$

(3) Let $I = \int x^3 . \ln 3x \, dx$

Let $u = \ln 3x$, $\therefore du/dx = 3/3x = \frac{1}{x}$
& $\frac{dv}{dx} = x^3$, $\therefore v = 3x^4$

So $I = 3x^4 \cdot \ln 3x - \int 3x^4 \cdot \frac{1}{x} \, dx$

$= 3x^4 . \ln 3x - 3\int x^3 \, dx$

$= 3x^4 \cdot \ln 3x - \frac{3}{4}x^4 + c$

(4) Let $I = \int x \cdot e^{-x} \, dx$

Let $u = x$, hence $du/dx = 1$
$\therefore \frac{dv}{dx} = e^{-x}$, hence $v = -e^{-x}$

So $I = -xe^{-x} - \left[\int 1 . (-e^{-x}) \, dx\right] = -xe^{-x} - e^{-x} + c$

(5) Let $I = \int 3x \cdot \sin x \, dx$

Let $u = 3x$, $\therefore \frac{du}{dx} = 3$

and $\frac{dv}{dx} = \sin x$, $\therefore v = -\cos x$

So $I = -3x \cos x - \left[\int -3 \cos x \, dx \right]$

$= -3x \cos x + 3 \sin x + c$

(6) Let $I = \int e^x . \sin 2x \, dx$

let $u_1 = e^x$, $\frac{du_1}{dx} = e^x$

$\frac{dv_1}{dx} = \sin 2x$, $v_1 = -\frac{1}{2} \cos 2x$

So $I = -\frac{1}{2} e^x \cos 2x - \left[\int -\frac{1}{2} e^x \cos 2x \, dx \right]$

$= -\frac{1}{2} e^x \cos 2x + \frac{1}{2} \int e^x . \cos 2x \, dx$

Now let $u_2 = e^x$, $\frac{du_2}{dx} = e^x$

and $\frac{dv_2}{dx} = \cos 2x$, $v_2 = \frac{1}{2} \sin 2x$

So $I = -\frac{1}{2} e^x \cos 2x + \frac{1}{2} \left[\frac{1}{2} e^x \sin 2x - \frac{1}{2} \int e^x \sin 2x \, dx \right]$

$$\therefore \quad I = -\frac{1}{2} e^{x} \cos 2x + \frac{1}{4} e^{x} \sin 2x - \frac{1}{4} I$$

$$\therefore \quad \frac{5}{4} I = -\frac{1}{2} e^{x} \cos 2x + \frac{1}{4} e^{x} \sin 2x + c$$

Hence $I = -\frac{2}{5} e^{x} \cos 2x + \frac{1}{5} e^{x} \sin 2x + c$

$$= \frac{1}{5} e^{x} (\sin 2x - 2\cos 2x) + c$$

7 Let $I = \int e^{2x} . \cos x \, dx$

Let $u_1 = \cos x$, $\therefore \frac{du_1}{dx} = -\sin x$

$\frac{dv_1}{dx} = e^{2x}$, $\therefore v_1 = \frac{1}{2} e^{2x}$

hence $I = \frac{1}{2} e^{2x} \cos x - \left[\int -\frac{1}{2} e^{2x} . \sin x \, dx \right]$

$$= \frac{1}{2} e^{2x} \cos x - \frac{1}{2} \int e^{2x} . \sin x \, dx$$

Now let $u_2 = \sin x$, $\therefore \frac{du_2}{dx} = \cos x$

$\frac{dv_2}{dx} = e^{2x}$, $\therefore v_2 = \frac{1}{2} e^{2x}$

$$\therefore \quad I = \frac{1}{2} e^{2x} \cos x - \frac{1}{2} \left[\frac{1}{2} e^{2x} . \sin x - \int \frac{1}{2} e^{2x} \cos x \, dx \right]$$

So $\quad I = \frac{1}{2} e^{2x} \cos x - \frac{1}{4} e^{2x} \cdot \sin x + \frac{1}{4} I$

$\therefore \quad \frac{3}{4} I = \frac{1}{2} e^{2x} \cos x - \frac{1}{4} e^{2x} \sin x + c$

So $\quad I = \frac{1}{3} e^{2x} (2 \cos x - \sin x) + c$

(8) Let $I = \int x^2 . e^{4x} \, dx$

Let $u_1 = x^2$, $\therefore \frac{du_1}{dx} = 2x$

& $\frac{dv_1}{dx} = e^{4x}$, $\therefore V_1 = \frac{1}{4} e^{4x}$

$\therefore \quad I = \frac{1}{4} x^2 . e^{4x} - \int 2x . \frac{1}{4} e^{4x} \, dx$

$= \frac{1}{4} x^2 . e^{4x} - \frac{1}{2} \int x . e^{4x} \, dx$

Now let $u_2 = x$, $\therefore \frac{du_2}{dx} = 1$

& $\frac{dv_2}{dx} = e^{4x}$, $\therefore v_2 = \frac{1}{4} e^{4x}$

So $\quad I = \frac{1}{4} x^2 . e^{4x} - \frac{1}{2} \left[\frac{1}{4} x e^{4x} - \int 1 . \frac{1}{4} e^{4x} \, dx \right]$

$= \frac{1}{4} x^2 . e^{4x} - \frac{1}{8} x e^{4x} + \frac{1}{32} e^{4x} + c$

(9) Let $I = \int e^{-x} \cdot \sin x \, dx$

Let $u_1 = \sin x$, $\therefore \frac{du_1}{dx} = \cos x$

& $\frac{dv_1}{dx} = e^{-x}$, $\therefore v_1 = -e^{-x}$

So $I = -e^{-x} \cdot \sin x - \left[\int -\cos x \cdot e^{-x} \, dx \right]$

$= -e^{-x} \cdot \sin x + \int e^{-x} \cdot \cos x \, dx$

Now let $u_2 = \cos x$, $\therefore \frac{du_2}{dx} = -\sin x$

& $\frac{dv_2}{dx} = e^{-x}$, $\therefore v_2 = -e^{-x}$

So $I = -e^{-x} \cdot \sin x + \left[-e^{-x} \cdot \cos x - \int (-e^{-x})(-\sin x) \, dx \right]$

$= -e^{-x} \cdot \sin x - e^{-x} \cdot \cos x - I$

So $2I = -e^{-x} \cdot \sin x - e^{-x} \cos x + k$

$\therefore$ $I = -\frac{1}{2} e^{-x} (\sin x + \cos x) + c$, where $c = k/2$.

(10) let $I = \int \ln 2x \, dx$

$= \int 1 . \ln 2x \, dx$

let $u = \ln 2x$, $\therefore \frac{du}{dx} = \frac{2}{2x} = \frac{1}{x}$

& $\frac{dv}{dx} = 1$, $\therefore v = x$

So $I = x \ln 2x - \int x . \frac{1}{x} \, dx$

$= x . \ln 2x - x + c$

(11) let $I = \int e^x . (x+1) \, dx$

could do $I = \int x . e^x + e^x \, dx = \int x . e^x \, dx + \int e^x \, dx$

& use The Standard Ans for $\int x . e^x \, dx$.

But here we will do it FROM scratch as follows:

let $u = x+1$, $\therefore \frac{du}{dx} = 1$

& $\frac{dv}{dx} = e^x$, $\therefore v = e^x$

So $I = e^x . (x+1) - \int e^x \, dx = e^x (x+1) - e^x + c$

$= x e^x + c$

(12) Let $I = \int x \cdot (1+x)^7 \, dx$

Let $u = x$, $\therefore \frac{du}{dx} = 1$

and $\frac{dv}{dx} = (1+x)^7$, $\therefore v = \frac{1}{8}(1+x)^8$

So $I = \frac{1}{8} x (1+x)^8 - \frac{1}{8}\int (1+x)^8 \, dx$

$= \frac{1}{8} x (1+x)^8 - \frac{1}{8} \cdot \frac{1}{9} (1+x)^9 + c$

$= \frac{1}{8} (1+x)^8 \left(x - \frac{1}{9} (1+x) \right) + c$

$= \frac{1}{8} \cdot \frac{1}{9} (1+x)^8 (8x - 1) + c$

(13) Let $I = \int x \cdot \sin\left(x + \frac{\pi}{6}\right) dx$

Let $u = x$, $\therefore \frac{du}{dx} = 1$

and $\frac{dv}{dx} = \sin\left(x + \frac{\pi}{6}\right)$, $\therefore v = -\cos\left(x + \frac{\pi}{6}\right)$

So $I = -x \cos\left(x + \frac{\pi}{6}\right) - \int -\cos\left(x + \frac{\pi}{6}\right) dx$

$= -x \cos\left(x + \frac{\pi}{6}\right) + \sin\left(x + \frac{\pi}{6}\right) + c$

(14) Let $I = \int x \cdot \cos nx \, dx$

let $u = x$, $\therefore \frac{du}{dx} = 1$

and $\frac{dv}{dx} = \cos nx$, $\therefore v = \frac{1}{n} \sin nx$

So $I = \frac{x}{n} \sin nx - \int 1 \cdot \frac{1}{n} \sin nx \, dx$

$= \frac{x}{n} \sin nx + \frac{1}{n^2} \cos nx + c$

(15) Let $I = \int x^n \cdot \ln x \, dx$

let $u = \ln x$, $\therefore \frac{du}{dx} = \frac{1}{x}$

& $\frac{dv}{dx} = x^n$, $\therefore v = \frac{x^{n+1}}{n+1}$

So $I = \frac{x^{n+1}}{n+1} \cdot \ln x - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} \, dx$

$= \frac{x^{n+1}}{n+1} \cdot \ln x - \frac{1}{n+1} \int x^n \, dx$

$= \frac{x^{n+1}}{n+1} \cdot \ln x - \frac{1}{(n+1)^2} \cdot x^{n+1} + c$

[OR let $u = x^n$ & $dv/dx = \ln x$; so $du/dx = nx^{n-1}$, $v = x\ln x - x$.

Hence $I = x^n (x \ln x - x) - n \int x^{n-1} (x \ln x - x) \, dx$

$= x^n (x \ln x - x) - nI + nx^n + c$

$\therefore (1+n) I = x(x \ln x - x) + nx^n + c \Rightarrow I = \frac{1}{1+n} (\ldots)$]

(16) Let $I = \int 3x \cdot \cos 2x \, dx$

Let $u = 3x$, $\therefore \frac{du}{dx} = 3$

& $\frac{dv}{dx} = \cos 2x$, $\therefore v = \frac{1}{2} \sin 2x$

So $I = \frac{3}{2} x . \sin 2x - \frac{3}{2} \int \sin 2x \, dx$

$= \frac{3}{2} x . \sin 2x + \frac{3}{4} \cos 2x + c$

(17) Let $I = \int 2 e^x \sin x \cos x \, dx$

Notice that $2 \sin x \cos x = \sin 2x$, so $I = \int e^x . \sin 2x \, dx$

So let $u_1 = e^x$, $\therefore \frac{du_1}{dx} = e^x$

& $\frac{dv_1}{dx} = \sin 2x$, $\therefore v_1 = -\frac{1}{2} \cos 2x$

So $I = -\frac{1}{2} e^x \cos 2x - \int -\frac{1}{2} e^x . \cos 2x \, dx$

$= -\frac{1}{2} e^x \cos 2x + \frac{1}{2} \int e^x \cos 2x \, dx$

Now let $u_2 = e^x$, $\therefore \frac{du_2}{dx} = e^x$

and $\frac{dv_2}{dx} = \cos 2x$, $\therefore v_2 = \frac{1}{2} \sin 2x$

So $I = -\frac{1}{2} e^x \cos 2x + \frac{1}{2}\left[\frac{1}{2} e^x \sin 2x - \frac{1}{2}\int e^x . \sin 2x \, dx\right]$

$= -\frac{1}{2} e^x \cos 2x + \frac{1}{4} e^x \sin 2x - \frac{1}{4} I + c$

$\therefore \quad 1\frac{1}{4} I = -\frac{1}{2} e^x \cos 2x + \frac{1}{4} e^x \sin 2x + c$

$I = -\frac{2}{5} e^x \cos 2x + \frac{1}{5} e^x \sin 2x + K \quad , \quad K = \frac{4}{5} c$

$= \frac{1}{5} e^x (\sin 2x - 2\cos 2x) + K$

Note that we could have done $I = \int (2e^x \sin x) . \cos x \, dx$

& taken $u = 2e^x . \sin x$

& $\frac{dv}{dx} = \cos x$ or vice versa.

However, the solution above is quicker & simpler.

(18) Let $I = \int x^2 . \sin x \, dx$

Let $u_1 = x^2$, $\therefore \frac{du_1}{dx} = 2x$

& $\frac{dv_1}{dx} = \sin x$, $\therefore v = -\cos x$

So $I = -x^2 \cos x - \int -2x \cos x \, dx$

$= -x^2 \cos x + 2\int x . \cos x \, dx$

Now let $u_2 = x$, $\therefore \frac{du_2}{dx} = 1$

& $\frac{dv_2}{dx} = \cos x$, $\therefore v = \sin x$

So $I = -x^2 . \cos x + 2\left[x . \sin x - \int 1 . \sin x \, dx \right]$

$= -x^2 \cdot \cos x + 2x \sin x + \cos x + c$

(19) let $I = \int e^{ax} . \sin bx \, dx$

let $u_1 = \sin bx$, $\therefore \frac{du_1}{dx} = b \cos bx$

and $\frac{dv_1}{dx} = e^{ax}$ & $v_1 = \frac{1}{a} e^{ax}$

So $I = \frac{1}{a} e^{ax} . \sin bx - \frac{b}{a} \int e^{ax} . \cos bx \, dx$

Now let $u_2 = \cos bx$, $\therefore \frac{du_2}{dx} = -b \sin bx$

& $\frac{dv_2}{dx} = e^{ax}$, $\therefore v_2 = \frac{1}{a} e^{ax}$

So $I = \frac{1}{a} e^{ax} . \sin bx - \frac{b}{a}\left[\frac{1}{a} e^{ax} \cos bx - \int -\frac{b}{a} e^{ax} . \sin bx \, dx \right]$

$= \frac{1}{a} e^{ax} . \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} I + c$